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THE LEAST DENSITY OF A SPHERICAL SWARM OF PARTICLES, WITH AN APPLICATION TO ASTRONOMICAL OBSERVATIONS OF K. KORDYLEWSKI

Frederick V. Pohle

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ABSTRACT

The least constant density of a spherical swarm of particles is first reviewed for the case of circular motion about a fixed attracting mass. The density ratio is defined as the ratio of the density of the swarm to the attracting mass divided by the volume of a sphere whose radius is the radius of the circular orbit. The classical linearized analysis shows that the density ratio must be at least three. This work is extended to take into account the finite size and the variable density of the swarm; the least density ratio is increased in each case. The problem is then generalized to the case where the swarm is located at either of the stable Lagrangian equilateral triangle points (L_4, L_5) in the restricted problem of three bodies. The least density ratio is now $3 - (9/4)\mu$ to first order terms in μ, and the mass is that of the combined earth-moon system; since μ is approximately 1/82 in the earth-moon system, the change in density from the classical value is slight.

The recent (1961) astronomical observations of K. Kordylewski of Krakow Observatory (Poland) with respect to the existence of a pair of cloud-like satellites near each stable L-point are discussed, and a dynamical model is analyzed which yields results that are in substantial agreement with the observations. The model assumes the existence of a small mass at either stable L-point and dust particles moving in the surrounding region. The main tool in the analysis is the Jacobi (energy) integral which determines the surfaces of zero relative-velocity; the model indicates that for an appropriate and plausible range of parameters the surfaces of zero relative-velocity can split into shapes that are similar to those noted by Kordylewski.

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INTRODUCTION

Studies in the nebular hypothesis and in the theory of comets have led to the consideration of the stability of a spherical swarm of particles in circular motion about a massive center of attraction (sun). Any particle of the swarm is attracted toward the sun, and toward the center of the swarm by particles interior to it. If it is assumed that the particles are at least 100 microns in diameter [1], we can restrict ourselves to gravitational forces alone and ignore radiation forces as well as electrical ones.

To illustrate the dynamical situation in the simplest way consider a pair of equal small particles ($m_1=m_2=m$) in circular motion about a fixed center of attraction M, ($m/M << 1$), as shown in Figure (1). For simplicity we assume that the orbits of m_1 and m_2 are in the same plane. The particles m_1 and m_2 can move in circular orbits of radii $(b-a)$ and $(b+a)$, if the sole force on either is due to M. The respective angular velocities of the radii vectors are then given by $\omega_1^2 = GM/(b-a)^3$ and $\omega_2^2 = GM/(b+a)^3$

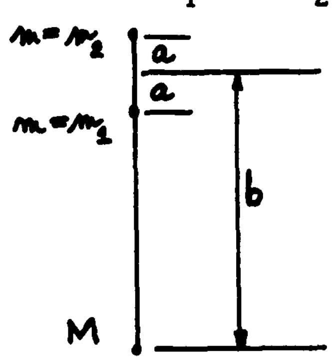

Figure 1

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.

where G is the gravitational constant. Thus $\omega_1 \neq \omega_2$ and in fact $\omega_1 > \omega_2$. If the particles are in the same radius at time $t = 0$, the inner particle will move more rapidly than the outer one and the particles will separate; the distance between them will be $> 2a$ and increasing until they are 180° apart.

However if the forces due to mutual attraction between m_1 and m_2 are included in the analysis, the force on m_1 will be diminished and that on m_2 will be increased at $t = 0$. This action will decrease ω_1 and increase ω_2 and it is possible to make $\omega_1 = \omega_2$ by proper choice of the parameters. It is only necessary to require that

$$GMm/(b-a)^2 - Gm^2/(4a^2) = m\omega^2(b-a),$$

$$GMm/(b+a)^2 + Gm^2/(4a^2) = m\omega^2(b+a),$$

if the particles are to remain on the same radius to M for all time. If we introduce a dimensionless ratio of frequent use, to be called the density ratio λ, as

$$\lambda = \left[\frac{(2m)}{(2a)^3}\right]\left[\frac{b^3}{M}\right] = \left[\frac{2m}{(4\pi/3)(2a^3)}\right]\left[\frac{(4\pi/3)b^3}{M}\right]$$

then elimination of ω between the two equations yields

$$\lambda = \frac{b^2(3b^2 + a^2)}{(b^2 - a^2)^2} . \tag{1}$$

The quantity λ is a ratio of densities. The first term is the total particle mass

(2m) divided by the volume of a sphere of radius (2a); the second term can be interpreted as the mass of the primary (M) divided by the volume of the orbit (radius b). If $b \gg a$, which means that the distance between the particles is small compared with b, then $\lambda = 3$ very nearly. The density ratio must therefore be 3 for the assumed motion to persist. If the value of λ is < 3 then the self attractive forces are insufficient to keep the particles together and the simple model shows that the density ratio of a swarm of particles must exceed a minimum value for the swarm to remain together.

The simple two-particle model is not realistic enough for the discussion of a swarm of particles more nearly spherical in distribution. In fact the assumed motion is unstable; the application of this result is of some interest in other problems (orbital rendezvous and docking dynamics of space vehicles) and is considered further in the Appendix.

The self-attractive forces in a spherical swarm are of a more complex nature than in the two-particle model; the force exerted on any particle of a spherical swarm with radially symmetric density, by the remainder of the swarm, will tend to zero as the particle approaches the center of the swarm. The self attractive force can become arbitrarily large in the two-particle model and this singularity makes the model unsuitable for further study of the stability of extended bodies composed of small particles.

The classical analysis of a spherically symmetric swarm of particles assumes that the radius of the swarm is infinitesimally small compared with the radius of the orbit and that the density is constant. This work is briefly reviewed in I and extended to the case of a finite swarm radius by means of an energy integral. The influence of variable density can also be taken into account in an approximate way under reasonable simplifications. Both effects increase the least density ratio.

It is of interest to extend the basic result of the linearized analysis to the case of motion with respect to two finite bodies in the sense of the restricted problem of three bodies. In this formulation the two finite bodies rotate uniformly in circles about their common mass-center and the third body is infinitesimal in mass. That is, the third body moves under the attractive forces of the two finite bodies, but does not influence their motion. The spherical swarm of particles is the third (infinitesimal) body.

It is well known, in the theory of the restricted problem of three bodies, that five equilibrium positions are possible, [2, chap. 8] . The three straight-line configurations (denoted by L_1, L_2, L_3) are unstable for any mass-ratio of the finite bodies. The two equilateral triangle solutions (denoted by L_4, L_5) are unstable if $\mu > 0.0385$ where μ is the ratio of the smaller finite mass to the total mass. The L_4, L_5 points are the vertices of equilateral triangles with the two finite masses at the remaining vertices and in the plane of the orbit. Since a small particle placed at L_4, L_5 in the earth-moon system ($\mu < 0.0385$) is at a position of stable equilibrium, at least according to the small displacement criterion of a linear analysis, it will tend to remain in this vicinity for a long time; ultimately the particle will leave the L_4, L_5 region, to be replaced by other particles that are moving in the surrounding space. It is therefore possible to have an accumulation of matter at the L_4, L_5 points.

However, the condensation of such matter into a stable swarm would again require considerations similar to that in I for a swarm moving in a circular orbit about a fixed mass. In the present case the motion of a swarm centered at L_4 or L_5 would again be circular about the mass-center of the two finite bodies. The precise considerations for this case are given in II for the case of a spherical swarm of constant density.

In the analyses outlined so far, the typical particle of the swarm was assumed to be internal to it. This is necessary for the discussion of the stability of the swarm and the determination of the least value of the density ratio λ . However, it is also necessary to consider a particle external to the swarm. This is the case in problems which involve the tail of a comet. Here we are concerned with particles external to the nucleus of a comet; the tail is assumed to be so tenuous that the particles in it do not attract each other but are influenced solely by the nucleus and the masses which can influence the motion of the nucleus.

These considerations bring us to the problem treated in III, in relation to the remarkable observations made at the Krakow Observatory by the Polish astronomer K. Kordylewski. In the search for additional natural satellites of the earth, Kordylewski looked for such objects near L_4 and L_5 . His observations indicate that two thin cloud-like satellites exist at each point. The problem treated in III is this: is it possible to explain these observations in terms of a simple dynamical model ?

The model assumes the existence of a small nucleus at either L point (L_4 or L_5) and the existence of tenuous matter near it. As a simplifying assumption the action of the moon is neglected and the nucleus is assumed to move in a circular orbit about the earth at a distance equal to that of the moon from the earth. It is a simple matter to derive a Jacobi integral for the particles external to the nucleus. If the velocity of the particles is very close to the orbital speed at that point (L), it can be shown that the surfaces of zero relative velocity become two small elongated tube-like surfaces near each L point. If the relative velocity increases, these surfaces are much enlarged physically or cease altogether, and the particles cannot remain near the nucleus for any length of time. Under appropriate conditions not only can the particles remain near the zero relative velocity surfaces, but if they do, they must do so near two small regions. Kordylewski's observations at each L point have indicated just such cloud-like formations; the observational problems are serious ones but observations [14, 15, 16, 17] have been announced for each L-point. Considerable work has also been done by others in the search for natural satellites which are much nearer to the earth. Here again the observational difficulties are great but of a different type; no announcements have been made as to the discovery of such near-earth natural satellites.

LIST OF MAIN SYMBOLS

$\underline{a}$		distance; orbital radius; semi-major axes
e		eccentricity of orbit (positive)
G		gravitational constant (units: $L^3M^{-1}T^{-2}$)
h		constant (twice rate of change of area swept out)
K		constant in Jacobi integral
k^2	=	GM/a^3 (units of k: T^{-1})
L		Lagrangian equilibrium point, in particular either L_4 or L_5
M, m		Masses
R, r		distances
u, v, w		coordinates
$X, Y; x, y, z$		coordinates
V		relative velocity in Jacobi integral
v		true anomaly
β		constant (mass/radius^3 of swarm)
γ		$(3\sqrt{3}/4)(1-2\mu)$
μ		ratio of smaller finite body to the sum of both finite bodies in the restricted problem of three bodies, $(0 < \mu \leq 1/2)$
λ		density ratio, or the ratio of the density of the swarm to the mass of the primary divided by the volume of a sphere whose radius is the radius of the orbit.
ξ, η		coordinates
τ	=	kt, dimensionless time
θ		angle
ρ, φ		polar coordinates
ω		angular speed
$\cdot$		d/dt (time differentiation)
$'$		$d/d\tau$ (dimensionless time differentiation)

The subject of interplanatary dust has been considered in several recent (and many past) studies, [1, 3, 4] . The present requirements are reasonable ones since such small dust-like particles are continuously moving in the spaces of the solar system. In fact the discussion of the phenomenon of the Gegenschein [2, p. 305] and [9] has some points of similarity to the present case even though the L_4, L_5 points are stable positions of equilibrium; in the Gegenschein problem the L points (L_1, L_2, L_3) are unstable positions of equilibrium. However, Kordylewski's observations indicate that a pair of cloud-like satellites do exist near each stable L-point. If it is assumed that some matter can accumulate at L_4 or L_5 in the form of a nucleus, the dynamical model can explain the form of the clouds observed by Kordylewski. Finally, the least density of the nucleus, considered as a swarm, is of possible interest in connection with space probes near the L_4, L_5 points; the actual density would seem to be comparable with the density of the atmosphere of the earth well inside the re-entry altitudes (less than 50 miles).

I. CIRCULAR MOTION ABOUT A FIXED MASS-POINT.

(a) Basic Equations; Linearized Analysis.

In Figure (2) consider the mass M to be fixed at the origin 0 of the (x, y, z) coordinate system. The center of the spherical swarm is at $C(\xi, \eta)$ in the $z = 0$ plane and point C moves uniformly in a circle of radius $\underline{a}$, where $a^2 = \xi^2 + \eta^2$, and r is the distance of any point P(x, y, z) of the swarm to the origin 0, where $r^2 = x^2 + y^2 + z^2$. The swarm is assumed to be spherical in shape and constant in density. If σ is the constant density of the swarm then the mass contained in a sphere of radius PC is $m = (4\pi/3)(PC)^3\sigma$ or $m/(PC)^3 = (4/3)\pi\sigma = \beta$ = constant.

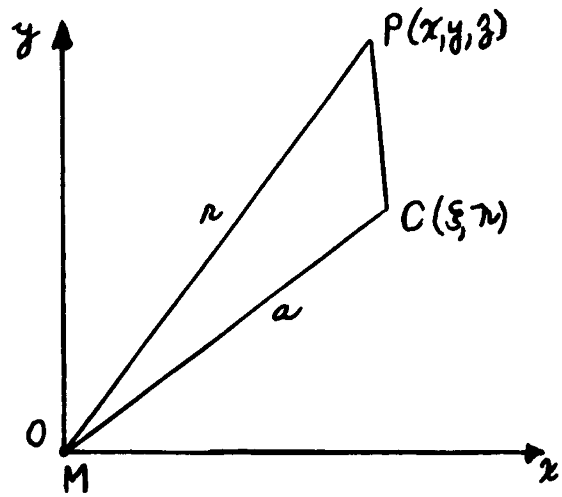

Figure 2

Under these assumptions it is easy to write the differential equations of motion of P with respect to the inertial system (x, y, z) as

$$\begin{aligned} \ddot{x} &= -GMx/r^3 - G\beta(x - \xi) \ , \\ \ddot{y} &= -GMy/r^3 - G\beta(y - \eta) \ , \\ \ddot{z} &= -GMz/r^3 - G\beta z \ , \end{aligned} \tag{2}$$

where $\cdot = d/dt$ and G is the gravitational constant.

The general equations (2) hold at $P = C$, that is, at $x = \xi$, $y = \eta$, $z = 0$ and $r = a$ = constant:

$$\ddot{\xi} = -GM\xi/a^3, \quad \ddot{\eta} = -GM\eta/a^3 . \tag{3}$$

If $k^2 = GM/a^3$ the equations (3) both assume the simple form $\ddot{\xi} + k^2\xi = 0$ and we choose

$$\begin{aligned} \xi &= a\cos(kt) , \\ \eta &= a\sin(kt) , \end{aligned} \tag{4}$$

as the coordinates of C as a function of time t . The equations (4) show that it is convenient to introduce $kt = \tau$ as the dimensionless time and a as the new unit of length; the dot ($\dot{}$) above a quantity denotes differentiation with respect to the time t and a prime (') above a quantity denotes differentiation with respect to the dimensionless time $kt = \tau$: $k(\dot{\ }) = (\ )'$. We also introduce new variables for (x, y, z) in a rotating reference system in which C is the fixed point at the x-axis at unit distance from 0 :

$$\begin{aligned} x &= a[(u+1)\cos\tau - v\sin\tau] \\ y &= a[(u+1)\sin\tau + v\cos\tau] \\ z &= aw . \end{aligned} \tag{5}$$

Thus u, v, and w are all small quantities compared with unity. If the substitutions (4) and (5) are made in Eqs. (2) and the trigonometric terms are eliminated the results can be written as

$$
\begin{aligned}
u'' - 2v' - u &= 1 - (u+1)/R^3 - \lambda u \ , \\
v'' + 2u' - v &= -v/R^3 - \lambda v \ , \qquad (6) \\
w'' \qquad\quad &= -w/r^3 - \lambda w \ ,
\end{aligned}
$$

if use is made of $k^2 = GM/a^3$, and $\lambda = [(4\pi/3)a^3/M]\sigma$ is the appropriate density ratio, and $R^2 = 1 + 2u + (u^2+v^2+w^2)$. In the linearized analysis, such as that outlined in Routh [6, p. 264-267] it is assumed that $R^{-2} = 1-2u$ and $R^{-3} = 1-3u$ and the linearized equations are then obtained in the form from Eqs. (6) as

$$
\begin{aligned}
u'' - 2v' + u(\lambda - 3) &= 0 \ , \\
v'' + 2u' + v(\lambda) &= 0 \ , \qquad (7) \\
w'' \ + \ w(1 + \lambda) &= 0 \ ,
\end{aligned}
$$

if squares of u, v or w are neglected in the expansions. The third of equations (7) has only oscillatory solutions since the density ratio is positive. If, in the first two equations we set $u = Ue^{\nu\tau}$ and $v = Ue^{\nu\tau}$, the condition on ν is

$$
\begin{vmatrix} [\nu^2 + (\lambda - 3)] & (-2\nu) \\ (2\nu) & [\nu^2 + \lambda] \end{vmatrix} = 0 \ , \qquad (8)
$$

or

$$\nu^4 + (2\lambda + 1)\nu^2 + (\lambda^2 - 3\lambda) = 0 \ .$$

For stability ν^2 must be negative; since

$$2\nu^2 = -(2\lambda + 1) \pm \sqrt{1 + 16\lambda} \ ,$$

the requirement is that $(1 + 16\lambda)$ be less than $(1 + 2\lambda)^2$ since λ is positive. This leads to the known condition

$$\lambda > 3 \tag{9}$$

which is the classical one discussed in Routh [6, p. 266].

According to Tisserand [5, p. 258ff], Schiaparelli obtained the criterion on statical grounds that if $\lambda < 2$ the swarm would dissolve due to the action of the attracting mass. Routh's statement [6, p. 264] that Schiaparelli required $\lambda > 2$ for stability is not quite accurate since only the weaker, but correct, assertion was made that the swarm would dissolve for $\lambda < 2$. The dynamical argument shows that the swarm would also dissolve if $\lambda < 3$.

Tisserand [5, p. 269ff] also discussed the dynamical arguments, which are due to Charlier and Picart; these are presented in simplified form in Routh [6, p. 264f]. The case where the orbit is non-circular was also noted briefly by Tisserand [5, p. 275] in connection with Floquet's theory of ordinary differential equations with periodic coefficients. Tisserand noted that if the eccentricity $\underline{e}$ of the orbit is small compared with unity, then $\lambda > 3 + He^2$ but he did not evaluate the constant H. Routh [6, p. 266] has given an approximate treatment to show that $H = 5$. The non-circularity of the orbit therefore raises the least value of the density ratio. Routh, [6, p. 406] has also discussed the equally important case where the swarm is non-spherical in shape but the orbit is circular. Picart [7] emphasized the importance of using the true anomaly as the new independent variable in those cases where the orbit was non-circular; see also [10]. Callandreau [8, 11, 12] also contributed to the study of this question. Some indication of the value of the transformations used by Picart will be noted briefly in IIc.

It is also possible to approach the problem of non-circular motion in a different manner although we shall not go into details. For example, we can assume that a spherical swarm is in arbitrary elliptical (parabolic, hyperbolic) motion about a fixed M; consider a particle on the surface of the swarm and acted upon by the particles internal to it, by M, and by assumed surface forces (tangential and normal) which constrain the particle to remain on the surface. Thus the motion of the particle is given and the equations of motion determine the value of the surface forces. The normal surface force can now be set equal to zero. In the case of a circular orbit this condition again leads to the requirement $\lambda = 3$ and this value increases with e.

(b) Jacobi (Energy) Integral; Arbitrary Swarm Radius.

The governing non-linear differential equations [6] can be integrated once if the equations are multiplied respectively by u', v', w' to get

$$(u')^2 + (v')^2 + (u^2+v^2)(\lambda - 1) - 2u - \frac{2}{\sqrt{(1+u)^2+v^2}} + 2 = \text{const} \tag{10}$$

where the constants have been appropriately adjusted and the dependence upon w has been assumed to be negligible for simplicity. The constant is zero if $u = v = 0$.

If $u = r\cos(\theta)$, $v = r\sin(\theta)$ and if we write

$$(u')^2 + (v')^2 + \Phi(r, \theta; \lambda) = \text{const.} \tag{11}$$

then $$\Phi(r, \theta; \lambda) = (\lambda - 1)r^2 - 2r\cos(\theta) - \frac{2}{\sqrt{1 + 2r\cos\theta + r^2}} + 2 . \tag{12}$$

The least value of λ is required such that $\Phi > 0$ for given (u, v) or (r, θ) , where $(0 < r < 1)$. For example if u, v are small and only quadratic terms are retained, $\Phi = (\lambda - 3)u^2 + v^2$ and $\Phi \geq 0$ if and only if $\lambda > 3$ for arbitrary u, v; this is the known linear criterion.

In the non-linear case we can first assume that r is a fixed quantity and seek to extremize Φ; this fixes θ as a function of r and three cases can arise: $\cos\theta = -r/2$ or $\sin\theta = 0$ ($\cos\theta = +1$ or -1) . The first case leads to $\Phi = \lambda r^2$ which does not restrict λ . In the two following cases the condition

$\cos\theta = -1$ leads to the larger value of λ in the form

$$\lambda = 3 + \frac{r(3-2r)}{(1-r)^2}$$

and this condition dominates the problem and Φ attains its minimum value under these conditions. If r is assumed to be negligibly small with respect to unity then the linear result $\lambda = 3$ is again obtained. Otherwise $\lambda = 3+3r$ to lowest order terms in r; in general the density ratio must increase if the swarm increases in size. This is a result to be expected on physical grounds and indeed we would expect the result to become unbounded as $r = 1$.

(c) Swarm of Variable Density.

Since the radius of the swarm is almost always small compared with the radius of the orbit, the effect of the finite size on λ is rather small. However, the effect of variable density is likely to be more marked. To take approximate account of this effect we return to the first two non-linear equations (6) and seek to modify the equations by a reasonable assumption about the variation of density with the radius of the swarm.

The quantity β has been defined as the mass of the swarm divided by the cube of the radius. If $\sigma = \sigma(r)$ is the density of the swarm as a function of r .

$$\beta = \frac{\int_0^r 4\pi r^2 \sigma \, dr}{r^3} .$$

If it is assumed that

$$\sigma = \sigma_o e^{-\gamma r^2} (1 - \frac{2\gamma}{3} r^2)$$

then
$$\beta = (4\pi\sigma_o/3)\, e^{-\gamma r^2} = \beta_o\, e^{-\gamma r^2}$$

The assumed density variation has been chosen conveniently to arrive at a simple form for $\beta(r)$; the density variation is reasonable since it is an exponentially decaying function of r, and of course it is necessary that r^2 be less than $3/(2\gamma)$.

The differential equations now take on the form

$$u'' - 2v' - (1+u) = - \frac{(1+u)}{\sqrt{(1+u)^2 + v^2}} - \beta_o\, e^{-\gamma r^2} ,$$

$$v'' + 2u' - v \quad = - \frac{v}{\sqrt{(1+u)^2 + v^2}} - \beta_o\, e^{-\gamma r^2} , \tag{13}$$

where $r^2 = u^2 + v^2$ and $rr' = uu' + vv'$. The Jacobi integral can be written as

$$(u')^2 + (v')^2 - (u^2+v^2+2u) - \frac{2}{\sqrt{(1+u)^2 + v^2}} - (\beta_o/\gamma)\, e^{-\gamma r^2} + (2 + \frac{\beta_o}{\gamma}) = \text{const} , \tag{14}$$

and an analysis similar to that for a finite radius shows that

$$\beta_o = \left[3 + \frac{r(3 - 2r)}{(1 - r)^2} \; e^{\gamma r^2} \right].$$

The effect of variable density is to increase the density ratio, which is again a result to be expected on physical grounds.

(d) Concluding Remarks

The dominant result of the simplest linear analysis is that the density ratio must exceed 3 for a stable swarm to exist in circular motion about a large attracting mass. The effects of finite size of swarm, eccentricity of orbit, and variable density all act to increase the least value of the density ratio. For the purposes of section II it is reasonable to restrict ourselves to the linearized analysis of a spherical swarm, but extend the analysis to the case of the restricted problem of three bodies. Again the orbit is circular and eccentricity effects can be neglected.

II: MOTION IN THE RESTRICTED PROBLEM OF THREE BODIES.

(a) Equations of Motion in Rectangular Coordinates.

The restricted problem of three bodies will be formulated in the usual way [2, chap. 8]; see Figure (3).

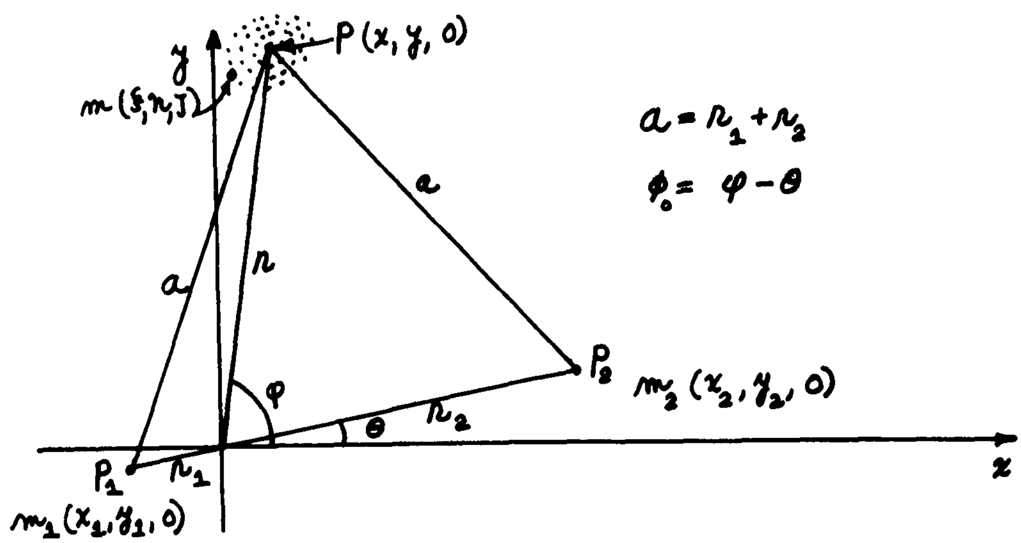

Figure 3

The plane of the orbit of the two finite bodies $m_1(x_1, y_1, 0)$ and $m_2(x_2, y_2, 0)$ is the plane $z = 0$ and their mass-center is always at the origin 0 of the inertial system. The center of the spherical swarm is at $P(x, y, 0)$ such that PP_1P_2 is an equilateral triangle. Thus P is at one of the stable equilibrium Lagrangian configurations in the restricted problem of three bodies. A general particle m of the swarm is at (ξ, η, ζ), near to P; the maximum radius of the swarm is small compared with the orbital radius $\underline{a}$. It is assumed of course that the mass ratio μ of the smaller mass to the total mass is small enough for P to be a stable point.

Simple geometric considerations show that

$$r\cos(\varphi_o) = (r_2 - r_1)/2 \ ,$$
$$r\sin(\varphi_o) = (a\sqrt{3})/2 \ , \quad (a = r_1 + r_2) \ , \tag{15}$$

and according to the assumptions stated we also require

$$x_1 = -(m_2/m_1)x_2, \quad y_1 = -(m_2/m_1)y_2 \ ;$$
$$m_2 = \mu M \ , \qquad m_1 = (1-\mu)M; \quad (\mu = m_2/M) \ , \tag{16}$$
$$r_1 = \mu a \ , \qquad r_2 = (1-\mu)a \ ,$$

The differential equations of motion of a particle inside the swarm are

$$\ddot{\xi} = -Gm_1(\xi - x_1)/r_1^3 - Gm_2(\xi - x_2)/r_2^3 - G\beta(\xi - x)$$
$$\ddot{\eta} = -Gm_1(\eta - y_1)/r_1^3 - Gm_2(\eta - y_2)/r_2^3 - G\beta(\eta - y) \tag{17}$$
$$\ddot{\zeta} = -Gm_1\zeta/r_1^3 \qquad - Gm_2\zeta/r_2^3 \qquad - G\beta\zeta$$

where
$$r_1^2 = (\xi - x_1)^2 + (\eta - y_1)^2 + \zeta^2$$
$$r_2^2 = (\xi - x_2)^2 + (\eta - y_2)^2 + \zeta^2 \tag{18}$$

If however the particle were external to the nucleus of radius $\underline{b}$ the last three terms on the right hand sides of equations (7) would be written, respectively, as

$$-\frac{(4\pi b^3\sigma/3)(\xi - x)}{r^3}, \quad -\frac{(4\pi b^3\sigma/3)(\eta - y)}{r^3}, \quad -\frac{(4\pi b^3\sigma/3)\zeta}{r^3}$$

where σ is the constant density and $r^2 = (\xi - x)^2 + (\eta - y)^2 + \zeta^2$.

Since the mass near P does not influence the orbital motion of m_1 and m_2 we have the equations of motion

$$\ddot{x}_2 = -Gm_1(x_2 - x_1)/a^3 \quad \text{and} \quad \ddot{y}_2 = -Gm_1(y_2 - y_1)/a^3 ,$$

where $a = r_1 + r_2$, and $M = m_1 + m_2$. Since $x_1 = -(\mu/1 - \mu)x_2$ and $y_1 = -(\mu/1 - \mu)y_2$ the result is that the differential equations can be written as

$$\ddot{x}_2 + k^2x_2 = 0, \quad \ddot{y}_2 + k^2y_2 = 0 ,$$

where $k^2 = GM/a^3$, and we assume that

$$\begin{aligned} x_2 &= r_2\cos kt = (1 - \mu)a\cos kt , \\ y_2 &= r_2\sin kt = (1 - \mu)a\sin kt , \\ x_1 &= -r_1\cos kt = -\mu a\cos kt, \\ y_1 &= -r_1\sin kt = -\mu a\sin kt . \end{aligned} \tag{19}$$

Since $x = r\cos(\theta + \varphi_o) = r\cos(kt + \varphi_o)$,

$$x = r(\cos kt)\cos(\varphi_o) - r(\sin kt)\sin(\varphi_o) \quad ,$$

$$x = (\cos kt)(r_2 - r_1)/2 - (\sin kt)(a\sqrt{3}/2) \quad . \tag{20}$$

The results for x and y are

$$x = a\left[\left(\frac{1-2\mu}{2}\right)\cos kt - \frac{\sqrt{3}}{2}\sin kt\right] \quad ,$$

$$y = a\left[\left(\frac{1-2\mu}{2}\right)\sin kt + \frac{\sqrt{3}}{2}\cos kt\right] \quad . \tag{21}$$

Finally we write

$$\xi = a\left[\left(u + \frac{1-2\mu}{2}\right)\cos kt - \left(v + \frac{\sqrt{3}}{2}\right)\sin kt\right] \quad ,$$

$$\eta = a\left[\left(u + \frac{1-2\mu}{2}\right)\sin kt + \left(v + \frac{\sqrt{3}}{2}\right)\cos kt\right] \quad , \tag{22}$$

$$\zeta = aw \quad ,$$

which introduces a uniformly rotating reference system; in this new system the finite masses $(\mu, 1-\mu)$ are fixed at the points on the u-axis at distances $(1-\mu, \mu)$, respectively, from the origin. In addition the coordinates have been chosen to make u and v small compared with unity since (ξ, η) and (x, y) are close to one another.

(b) Stability Criterion.

The same assumptions will be made here that were made in the linearized analysis of I, that is, in the equations

$$u'' - 2v' - u = (1 - 2\mu)/2 - \frac{(1-\mu)(u+1/2)}{p^3} - \mu \frac{(u - 1/2)}{q^3} - \lambda u \quad ,$$

$$v'' + 2u' - v = (\sqrt{3}/2) - \frac{(1-\mu)(v + \sqrt{3}/2)}{p^3} - \mu \frac{(v + \sqrt{3}/2)}{q^3} - \lambda v \quad , \qquad (23)$$

$$w'' = -(1-\mu)w/p^3 - \mu w/q^3 - \lambda w \quad ,$$

where
$$p^2 = (u + 1/2)^2 + (v + \sqrt{3}/2)^2 + w^2 \quad ,$$
$$q^2 = (u - 1/2)^2 + (v + \sqrt{3}/2)^2 + w^2 \quad , \qquad (24)$$

it is assumed that

$$p^{-3} = 1 - (3/2)(u + v\sqrt{3}) \quad ,$$
$$q^{-3} = 1 - (3/2)(-u + v\sqrt{3}) \quad , \qquad (25)$$

to lowest order terms. If terms quadratic in u, v, and w are neglected in the expansions the equations can be written as

$$u'' - 2v' + (\lambda - 3/4) - \gamma v = 0 \ ,$$
$$v'' + 2u' + (\lambda - 9/4) - \gamma u = 0 \ , \qquad (26)$$
$$w'' + (1 + \lambda)w \qquad = 0 \ ,$$

where $\gamma = (3\sqrt{3}/4)(1-2\mu)$; the last equation again has only oscillatory solutions and is independent of the first two equations. If in the first two equations it is assumed that $u = Ue^{\nu\tau}$, $v = Ve^{\nu\tau}$ the characteristic equation is determined by

$$\begin{vmatrix} (\nu^2 + \lambda - 3/4) & -(2\nu + \gamma) \\ (2\nu - \gamma) & (\nu^2 + \lambda - 9/4) \end{vmatrix} = 0 \ .$$

Again ν^2 must be negative for stability, and steps similar to that used in I lead to the requirement that

$$\lambda > (3/2) + \sqrt{(\gamma^2 + 9/16)} \ .$$

If $\mu = 0$, $\gamma^2 = 27/16$ and $\lambda > 3$, as in I . Since μ is small in the earth-moon system (approximately 1/82) and must be less than 0.0385 for stability at the L_4, L_5 points in any case, it can be assumed that μ is small compared with unity. If first order terms in μ are retained the result is that

$$\lambda > 3 - (9/4)\mu \ .$$

The least value of the density ratio is therefore changed by about one per-cent in the case of the earth-moon system.

(c) Concluding Remarks

(i) The stability criterion, based upon a linearized analysis, shows that the least density ratio of the spherical swarm is $3-(9/4)\mu$. In the presence of a single body the criterion is $\lambda > 3$ but it must be recalled that the masses used for comparison are not the same in both cases. In the latter case we have

$$\lambda_1 = \sigma_s/[M_E/(\tfrac{4}{3}\pi a^3)] > 3 \quad ,$$

where M_{eE} is the mass of the earth and σ_s = density of the swarm . In the former case we have

$$\lambda_2 = \sigma_s/[(M_E + M_m)/(\tfrac{4}{3}\pi a^3)] > 3-(9/4)\mu \quad ,$$

where M_m is the mass of the moon. Comparison of the two density ratios shows that $\lambda_2 = \lambda_1(1-\mu)$,

or

$$\lambda_1 > 3+(3/4)\mu \quad ,$$

in terms of the mass of the earth. Since the effect is small in either case we evaluate the density $3M/(4\pi a^3/3) = [3M/(4\pi R^3/3)][(R/a)^3]$ where M is the mass of the earth, R is the radius of the earth and $\underline{a}$ is the lunar distance.

Since the first factor is the mean density of the earth (5.5 gm/cm^3) and the second factor (a/R) is approximately 60, the density is approximately (1/13,000) grams/cm^3 . This is the least density of a spherical swarm at the lunar distance, or one located at either L-point. The density of the atmosphere at sea-level is approximately (1/800) grams/cm^3 and is decreased by a factor of 10 for each 12 miles increase in altitude, roughly [13]. Thus the density at 12 miles would be (1/8000) grams/cm^3 and (1/80,000) grams/cm^3 . The least density of the swarm would therefore be equivalent to the density of the earth's atmosphere in the range 12-24 miles above sea-level, or well within the re-entry altitudes.

(ii) Although an explicit Jacobi integral does not exist if $e \neq 0$ in the restricted problem of three bodies, it is nevertheless possible to write as follows: The same general notation is adopted as in (IIa), but now $r_2 = r = \frac{a(1-e^2)}{1+e\cos v}$ $(0 \leq e^2 < 1)$; v is the true anomaly, and

$$x_2 = r\cos v \;; \qquad x_1 = -[(\mu r)/(1-\mu)]\cos v \;; \qquad r^2\dot{v} = h \;,$$

$$y_2 = r\sin v \;; \qquad y_1 = -[(\mu r)/(1-\mu)]\sin v \;; \quad h^2 = GMa(1-e^2)(1-\mu)^3 \;.$$

The true anomaly v is now used as the independent variable in place of the time t . In the case of a circular orbit the differentiations with respect to these quantities differ only by a constant but in the present case this is not so. Since $\dot{r} = (eh)(\sin v)/[a(1-e^2)]$ and $\dot{v} = h/r^2$, we require the change from time differentiation (denoted by a dot) to differentiation with respect to the true anomaly (denoted by a prime). If f is a given function we have $\dot{f} = \dot{v}f'$ and $\ddot{f} = f'\ddot{v} + f''(\dot{v})^2$; with the previous results this leads to

$$\ddot{f} = \frac{-2eh^2 \sin v}{a(1-e^2)r^3} f' + \frac{h^2}{r^4} f''$$

Non-uniformly rotating coordinates are now introduced by means of the substitutions

$$\xi = X\cos v - Y\sin v \quad ,$$

$$\eta = X\sin v + Y\cos v \quad ,$$

$$\zeta = Z \quad .$$

The finite masses are always on the X-axis, but do not remain at the same locations. Finally, since r is now a variable we introduce new distances so that the instantaneous distance between m_1 and m_2 is the unit of distance; this changes with time of course and we write

$$X = xr, \qquad Y = yr, \qquad Z = zr \quad ,$$

where (x, y, z) are the new dimensionless coordinates. In the subsequent transformations we use the facts that $r'/r^2 = \frac{e \sin v}{a(1-e^2)}$ and $\frac{r''}{r^2} - \frac{2(r')^2}{r^3} = \frac{1}{r} - \frac{1}{a(1-e^2)}$.

The ultimate result is that

$$(1 + e\cos v)[(x')^2+(y')^2+(z')^2]' = (x^2+y^2)' - (e\cos v)(z^2)' + [\frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2}]'$$

where
$$\rho_1^2 = (x+\mu)^2 + y^2 + z^2 \;,$$
$$(\rho_{1,2} > 0)$$
$$\rho_2^2 = (x - [1-\mu])^2 + y^2 + z^2 \;,$$

Clearly if $e = 0$ the differentiations with respect to the true anomaly (′) and those with respect to the time (˙) are proportional to one another and the result above reduces to the usual form of the Jacobi integral (2, p. 281) .

III: DYNAMICAL MODEL FOR THE CLOUD-SATELLITES AT THE L_4 AND L_5 POINTS OBSERVED BY K. KORDYLEWSKI (KRAKOW OBSERVATORY).

(a) Observations of K. Kordylewski

Three recent issues (1961) of "Sky and Telescope" have announced the observations of K. Kordylewski with respect to cloud-like satellites at the L_4 and L_5 points, [14, 15, 16, 17]. In the first of the issues [14] a pair of cloud-like satellites was observed near the L_5 point, which is the trailing point 60° behind the moon. The two libration clouds were both near L_5, several degrees apart and it was suggested that similar clouds could be found near L_4. The brief announcements in the first issue were supplemented in the August 1961 issue [15] where some of the observational difficulties were noted. The luminous patches appeared to be at least two degrees in diameter; two clouds about eight degrees apart were identified and it was also suggested that large telescopes be used to locate individual meteoroids in the libration clouds. The third note, in the December 1961 issue [16] confirmed the existence of such clouds near the L_4 point as well. The preliminary results indicate that the L_4 object was detected as a pair of dim spots, each about five degrees in diameter and nearly touching. Some direct measurements were given in [17] by K. Kordylewski.

(b) Basic Equations for the Dynamical Model.

The dynamical model has been outlined in the Introduction; this consists in the assumption of a small mass at or near to one of the L-points (L_4 or L_5)

and tenuous matter (dust) in the surrounding space. It will be assumed that the nucleus as well as the cloud near it move in a circular orbit about the earth at a distance equal to the lunar distance. The effect of the moon will be ignored in order to simplify the presentation. Since the sphere of activity of the moon [18] is too small to enclose either L-point it is reasonable to neglect the effect of the moon. More precise calculations show that the effect of the moon is slight in this case, although the moon does of course determine the location of the L-points and the fact that they are stable equilibrium points in the earth-moon system.

The small mass of the nucleus is $m(x, y)$ in Figure (4); $\underline{a}$ is the lunar distance and the plane of the orbit of m is the (x, y) plane. A general particle of the cloud is at (ξ, η, ζ) and the distance from the nucleus to any particle is r_2 which is small compared with $\underline{a}$ and r_1, the distance to the earth of mass M at the origin; the ratio $m/M = \beta$ is assumed to be much less than unity.

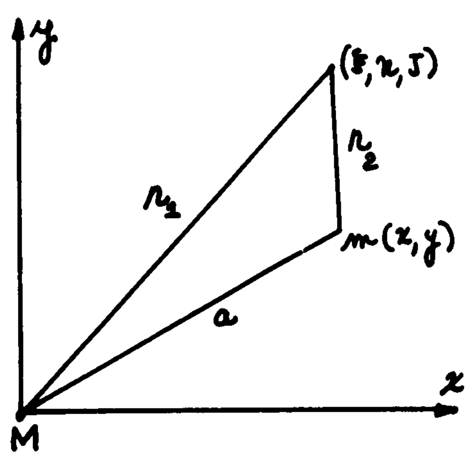

Figure 4

The particle (ξ, η, ζ) is near m but outside the radius of nucleus. The general particle is attracted by m and by M but does not influence their motion; m is in circular motion about the fixed mass M. The equations of motion can be written as

$$\ddot{\xi} = -GM\xi/r_1^3 - Gm(\xi-x)/r_2^3 ,$$

$$\ddot{\eta} = -GM\eta/r_1^3 - Gm(\eta-y)/r_2^3 , \tag{27}$$

$$\ddot{\zeta} = -GM\zeta/r_1^3 - Gm\zeta/r_2^3 ,$$

where G is the gravitational constant, and

$$r_1^2 = \xi^2 + \eta^2 + \zeta^2 ,$$

$$r_2^2 = (\xi-x)^2 + (\eta-y)^2 + \zeta^2 ,$$

$$x = a\cos(kt), \quad y = a\sin(kt); \qquad k^2 = GM/a^3 .$$

A rotating coordinate system is now introduced in which m is a fixed point on the X-axis; the unit of distance is chosen to be a and the dimensionless time $\tau = kt$ is introduced as the new independent variable, with $' = d/d\tau$:

$$\xi = a[X \cos\tau - Y \sin\tau] ,$$

$$\eta = a[X \sin\tau - Y \cos\tau] , \tag{28}$$

$$\zeta = aZ .$$

In addition we have that $m/M = \beta$ and

$$R_1^2 = X^2 + Y^2 + Z^2 ,$$

$$R_2^2 = (X-1)^2 + Y^2 + Z^2 , \quad (\beta, R_1, R_2 > 0) .$$

The final forms of the differential equations are

$$
\begin{aligned}
X'' - 2Y' - X &= -X/R_1^3 - \beta(X-1)/R_2^3 \ , \\
Y'' + 2X' - Y &= -Y/R_1^3 - \beta Y/R_2^3 \ , \qquad (29) \\
Z'' \qquad\qquad &= -Z/R_1^3 - \beta Z/R_2^3 \ .
\end{aligned}
$$

(c) Jacobi Integral; Discussion of Zero Relative Velocity Surfaces.

If the equations (29) are multiplied by X', Y', Z' respectively and integrated the result is the Jacobi integral in the form

$$V^2 = (X')^2 + (Y')^2 + (Z')^2 = X^2 + Y^2 + \frac{2}{R_1} + \frac{2\beta}{R_2} - K' \ , \qquad (30)$$

where K' is a positive constant in most further discussions. The discussion of the possible zero relative velocity surfaces is simplified if cylindrical coordinates are introduced:

$$
\begin{aligned}
Z &= 1 + \rho\cos\varphi \ , \\
Y &= \rho\sin\varphi \ , \qquad (31) \\
Z &= z \ .
\end{aligned}
$$

Terms up to $O(\rho^2)$ will be retained and particular attention will be paid to the value $z = 0$ and to those values of z that are small compared with the radial distance ρ .

The Jacobi integral can be written as

$$V^2 = 3\rho^2\cos^2\varphi + \frac{2\beta}{\sqrt{\rho^2+z^2}} - (K' + z^2) , \tag{32}$$

where $K' + z^2$ will be denoted by K, a constant for constant z. In particular if $z = 0$ the Jacobi integral can be written as

$$V^2 = 3\rho^2\cos^2\varphi + \left(\frac{2\beta}{\rho}\right) - K , \tag{33}$$

which is the form which will be used most frequently. If a particle is assumed to be at a given point (ρ_o, φ_o) with a given speed then the constant K is fixed; if V is zero in Eq. (33) the curves will be the traces of the zero-V surfaces $(z = 0)$, if they exist.

The dynamical equations of motion can also be written in terms of the cylindrical coordinates. If simplications are introduced which are consistent with Eq. (32) the equations can be written as

$$\begin{aligned} \rho'' - \rho(\varphi')^2 - 2\rho\varphi' &= 3\rho\cos^2\varphi - \frac{\beta\rho}{(\rho^2+z^2)^{3/2}} , \\ \rho\varphi'' + 2\rho'\varphi' + 2\rho' &= -3\rho\sin\varphi\cos\varphi , \\ z'' &= -z - \frac{\beta z}{(\rho^2+z^2)^{3/2}} , \end{aligned} \tag{34}$$

where $V^2 = (\rho')^2 + (\rho\varphi')^2 + (z')^2$. However if $(z/\rho) << 1$ the right hand sides of Eqs. (34) can be replaced by $3\rho\cos^2\varphi - (\beta/\rho^2)$; $-3\rho\sin\varphi\cos\varphi$; $-z$, respectively. In the latter case the last equation of (34) is $z'' + z = 0$ which has only oscillatory solutions and so the motion of z is known.

The dynamical equations (34) of course determine the actual motion of each particle if the initial conditions are prescribed. However much more interest is centered on the Jacobi integral (33) which yields a relationship between the (relative) position coordinates and the speed in any realizable motion. That is, if $K > 0$ it is possible for V to equal zero by proper choice of (ρ, φ); if V_o^2, the square of the initial speed, is sufficiently large then K is negative and it is not possible for V to vanish. Thus in the latter case no surfaces of zero relative velocity exist but in the former case $(K > 0)$ they do exist. Our object is to study such surfaces, or more precisely stated, the traces of such surfaces $(z = 0)$.

It may be assumed that K has been determined as

$$K = 3\rho_o^2 \cos^2\varphi + (2\beta/\rho_o) - V_o^2 \geq 0 \quad , \tag{35}$$

and that we have set $V = 0$ to obtain the equation

$$3\rho^3 \cos^2\varphi - K\rho + 2\beta = 0 \quad . \tag{36}$$

The cubic equation in ρ has one real negative root since the leading coefficient and the constant term are positive; this root is of no physical interest. The remaining two roots are either conjugate complex roots or both positive, since the sum of all roots is zero; our interest is in the latter case, which requires that

$$\cos^2\varphi < K^3/(81\beta^2) \quad . \tag{37}$$

In the form of the inequality (37), two cases of immediate interest arise. In the first, if $K^3 > 81\beta^2$, then no restriction is placed on φ, since the inequality is always satisfied; it is assumed of course that K must be positive in all cases. In the second case, if $K^3 < 81\beta^2$, then there is a restriction on φ and not all angles are possible; the admissible regions for φ are shown shaded in the sketch below, Figure (5).

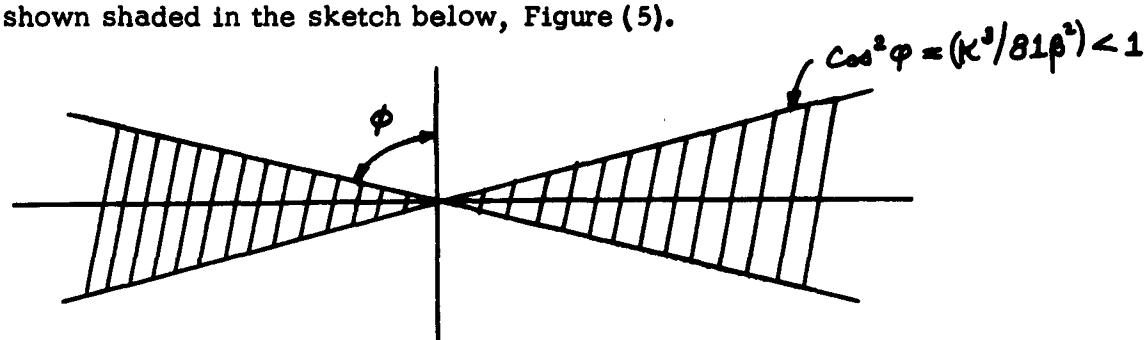

Figure 5

If $K^3 > 81\beta^2$, all angles are admissible; in particular the limiting case is reached if $K^3 = 81\beta^2$; if $\cos^2\varphi = 1$ the cubic equation can be written as

$$3R^3 - R + \gamma = 0 \quad ,$$

where $\rho = R\sqrt{K}$ and $\gamma^2 = 4\beta^2/K^3 = 4/9$. The cubic $3R^3 - R + (2/9) = 0$ has the repeated root $R = +(1/3)$ or $\rho = (\sqrt{K})/3$. The traces in this case are shown in the sketch below, Figure (6), and are denoted by C_1.

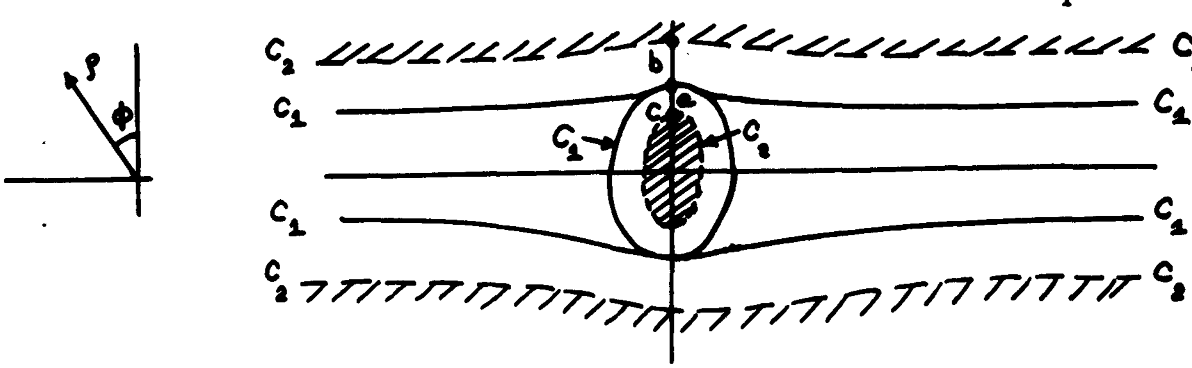

Figure 6

If K^3 is larger than $81\beta^2$ the traces are indicated by C_2; for such values of K (β fixed) there is a closed inner oval and a pair of curves which extend to infinity and reach the asymptotic width $2\sqrt{(K/3)}$ in all cases. It should be recalled that the unit of distance is the lunar distance <u>a</u> . Thus as K increases the inner oval becomes smaller in size and the outer traces move outward; the asymptotic width increases with K . The regions of positive V^2 are outside the pair of curves and inside the oval. These regions are indicated for C_2 by cross-hatched lines. The critical distance along $\varphi = 0$ occurs for C_1; $\rho^2 = K/9 = (3^{1/3}/9)\beta^{2/3} = \rho^2_{crit} = \rho^2_c$. For example, if a particle is placed ($\varphi = 0$) at the distance ρ with $V_o = 0$, then if $\rho^2 < \rho^2_c$ the particle must remain inside the oval, but if $\rho^2 > \rho^2_c$ the particle must remain in the region external to the outer traces and cannot penetrate to the L-region. A particle placed at <u>a</u> in Figure (6) with $V_o = 0$ will remain there according to the equations of motion; a particle placed at <u>b</u> will move outward but a particle placed at <u>c</u> will move inward ($V_o = 0$) . If the initial speed V_o is not zero then a similar analysis can be made if we require that K^3 be at least $81\beta^2$.

If however V_o is large enough it is possible for K^3 to decrease and remain positive so that K^3 is less than $81\beta^2$ but greater than zero. The traces then change in form and only certain ranges of φ are admitted, as shown in a typical case in the sketch, Figure (7).

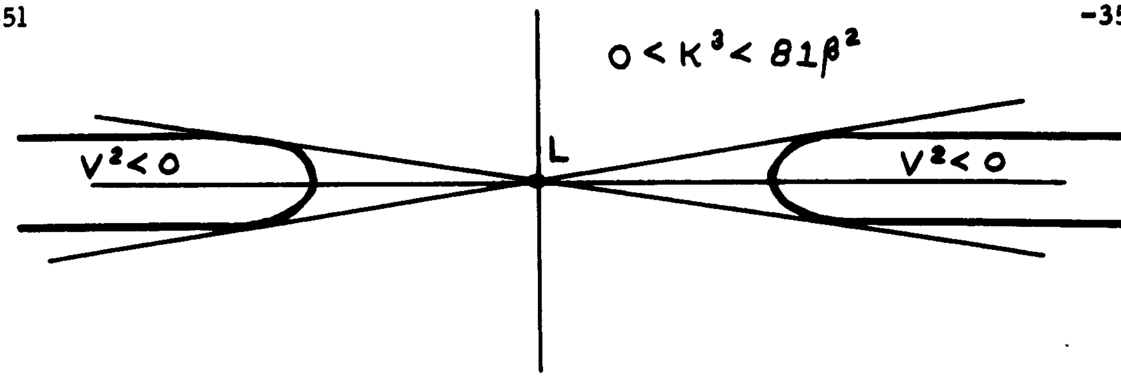

Figure 7

The regions of negative V^2 are inside the branches and the particles cannot enter such spaces. The asymptotic width of the branches is still $2\sqrt{(K/3)}$ but now K is smaller than in the previous case and so the asymptotic width is much narrower.

In the present case consider a particle in the admissible region with $V_o^2 > 0$; see Figure (8). Since $V^2 = 3\rho^2 \cos^2\varphi + \frac{2\beta}{\rho} - K$, $K = 3\rho_o^2 \cos^2\varphi + \frac{2\beta}{\rho_o} - V_o^2$. At point P, K is fixed by the initial coordinates and speed (energy). If in Figure (8) the particle moves toward Q, V will decrease and similarly for the points R, S, ... which approach the curve of zero relative velocity.

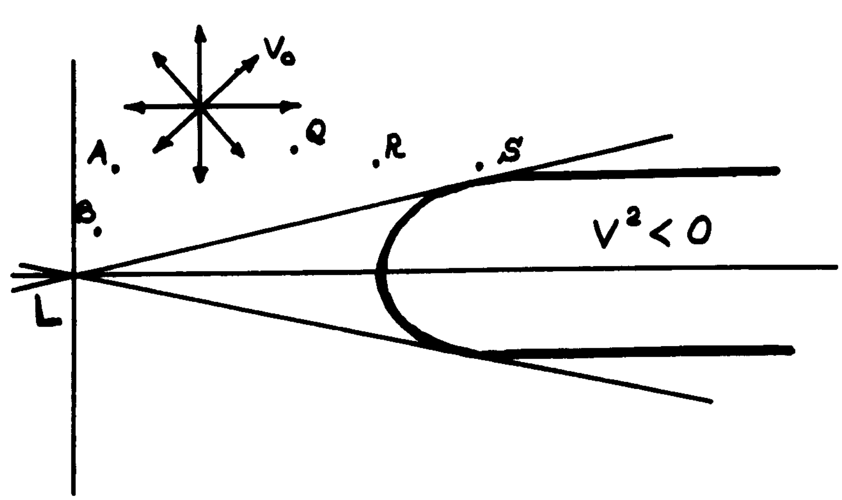

Figure 8

The equations of motion (34) of course determine the actual path in the relatives coordinates. The acceleration is not zero on the zero relative velocity curves, but is normal to the curves and directed outward. If some particles move toward A, B, ... then clearly $(2\beta/\rho)$ will increase and so will V^2 and the particle can move rapidly toward the L-point which is the origin of the coordinate system. The essential point is that some particles may approach to the $V = 0$ curves and remain there for an appreciable time and a slow accumulation of matter can take place near such curves. If K is smaller at P than before, the curve $V = 0$ is thinner and inside the one shown in the sketch above, Figure (8). The intersection point with the axis is $2\beta/K$ and this point moves outward. Thus a narrow region of space $(z = 0)$ acts as an area where particles may move slowly and accumulate. Ultimately these particles will drift into other regions of space but since there is a steady drift of such particles throughout the space there will always be new particles streaming into the region to replace those which may have moved away. This requires that the particle speeds be close to the orbital speeds at the L-points since K is small (small relative velocity). If the initial speeds are large enough then no zero V surfaces can exist; this is reasonable for it indicates that the particles cannot be trapped, or brought near to rest, in the vicinity of the L-point. This is the case if $V_o^2 > 3\rho_o^2 \cos^2\varphi + \frac{2\beta}{\rho_o}$, and therefore the right hand expression can be considered to yield a minimum escape speed in the region near the L-points.

If z is not zero, but still small compared with ρ, then the constant in Eq. (32) is merely increased slightly. The discussion is essentially the same as before, but now we require that $(K + z^2)^3 > 81\beta^2$ and this also limits the region in the z-direction. It is problematical if this knowledge could be exploited observationally to determine an approximate value of β; it is not likely that the observations are clearly enough defined for this to be the case, or if the form of the clouds is in fact constant in time.

The equations of motion, Eqs. (34), could be used to investigate further the particle motions near the zero-V surfaces; a simplified analysis, which is omitted, indicates that the particles will tend to move away from such surfaces $(V = 0)$ more quickly near the L-points and less so away from the L-points.

(d) Concluding Remarks

If the dynamical model is assumed to consist of a nucleus surrounded by dust particles moving near the L-regions then the main tool in the investigation of the possible regions where such particles may migrate is provided by the Jacobi integral. The constant K is fixed by the initial conditions. If K is negative then V_o is so large that no zero V surfaces are possible and no particle can come to rest near L (unless it strikes and is absorbed by the nucleus). If V_o is sufficiently small and K^3 exceeds $81\beta^2$ then we have the case shown in Figure (6), but if K^3 is less than $81\beta^2$ we have the case shown in Figure (7). The asymptotic width of the curves is proportional to $\sqrt{K}$ and this means that the curves are much thinner in the latter case, which is the one where the zero-V curves split into two narrow branches. Although in the former case a single surface is possible this is much wider and the accumulation would not be confined to successively narrower regions. The dynamical model assumed therefore gives rise to the possibility that faint cloud-like patches may appear in the region of the L-points.

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APPENDIX: STUDY OF THE STABILITY AND MOTION OF THE TWO-PARTICLE MODEL.

In the Introduction the circular motion of two equal infinitesimal particles of mass m was considered in relation to a fixed mass M . The value of $\lambda = mb^3/(4Ma^3)$ is 3 if (a/b) is much less than unity. If, for example, m = 2500kg, then 2m = 5000kg or about 11,000 pounds. If two such masses are in close circular orbit about the earth it is interesting to compute how close two such bodies must be for β to equal 3 . If the distance b is kR , where k is a constant multiplier and R is the radius of the earth, and use is made of the fact that the mean density of the earth is 5.5 gm/cm^3 then the distance between the two particles (2a) is given by 2a = 41.5k in centimeters for the given data (m = 2500kg) . Thus at a distance of 10 earth radii (40,000 miles), 2a = 415cm or 4.15 meters or approximately 13 feet . These figures would change if the distances and densities would be those for the moon, since the density is lower and the radius is smaller.

The question of the stability of such an assumed motion can be investigated in a simplified way if we restrict ourselves to the orbital plane of motion; see Figure (9).

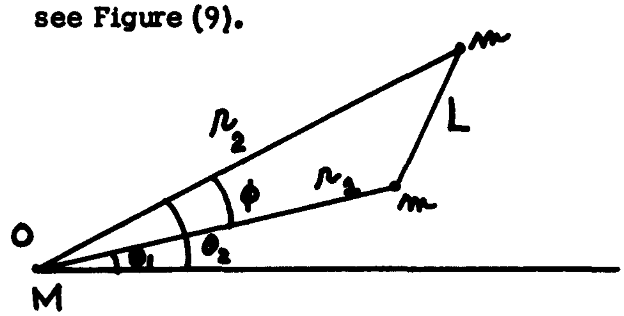

Figure 9

$$L^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi), \quad (L > 0) ,$$

$$\varphi = (\theta_2 - \theta_1) .$$

The equations of motion in polar coordinates are

$$
\left\{
\begin{aligned}
\ddot{r}_1 - r_1\dot{\theta}_1^2 &= -GM/r_1^2 - (Gm/L^3)(r_1 - r_2\cos\varphi) \;, \\
r_1\ddot{\theta}_1 + 2\dot{r}_1\dot{\theta}_1 &= (Gm/L^3)(r_2\sin\varphi) \;;
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\ddot{r}^2 - r_2\dot{\theta}_2^2 &= GM/r_2^2 - (Gm/L^3)(r_2 - r_1\cos\varphi) \;, \\
r_2\ddot{\theta}_2 + 2\dot{r}_2\dot{\theta}_2 &= (Gm/L^3)(r_1\sin\varphi) \;.
\end{aligned}
\right.
\tag{38}
$$

In the case of uniform circular motion it is assumed that

$$
\left\{
\begin{aligned}
r_1 &= R_1 = \text{constant}, \\
r_2 &= R_2 = \text{constant} \;,
\end{aligned}
\right.
\qquad
\theta_1 = \theta_2 = (\omega t) \;, \qquad \omega = \text{constant} \;,
\tag{39}
$$

which implies that $\varphi = 0$ and $L = |R_2 - R_1| = d > 0$; the second and fourth equations in Eqs. (38) are identically satisfied and elimination of ω^2 between the two remaining equations leads to

$$
\frac{m}{M} = \frac{d^3(R_2^2 + R_1R_2 + R_1^2)}{(R_1R_2)^2\,(R_1 + R_2)}
\tag{40}
$$

which is equivalent to the relation (1) . The following two expressions for ω^2 will be of use in subsequent work:

$$\omega^2 = (GM/R_1^3) - (Gm/d^3)(R_2-R_1)/R_1 \ , \tag{41}$$

$$\omega^2 = (GM/R_2^3) + (Gm/d^3)(R_2-R_1)/R_2 \ . \tag{42}$$

In the slightly disturbed motion it is assumed that each of the 4 coordinates may be varied; that is,

$$\left\{\begin{array}{l} r_1 = R_1(1+\rho_2) \ , \\ \\ r_2 = R_2(1+\rho_2) \ , \end{array}\right. \qquad \left\{\begin{array}{l} \theta_1 = \omega t + \alpha_1 \ , \\ \\ \theta_2 = \omega t + \alpha_2 \ , \end{array}\right. \tag{43}$$

where $\rho_1, \rho_2, \alpha_1, \alpha_2$ are dimensionless quantities. It is also assumed that φ is so small that $\cos\varphi = 1$ and that $\sin\varphi = \varphi = \alpha_2 - \alpha_1$; only terms linear in ρ, α will be retained in the stability analysis. Finally we set $\omega t = \tau$ with $'$ denoting $d/d\tau$, and approximate the binomial expansions in the usual way: $r_1^{-2} = R_1^{-2}(1-2\rho_1)$ and similarly for subscript 2, and

$$L^{-3} = d^{-3}\left\{1 - 3\left(\frac{R_2\rho_2 - R_1\rho_1}{d^2}\right)\left(R_2 - R_1\right)\right\} \ .$$

The first dynamical equation can be written in the following way, after simplifications have been made:

$$\omega^2\left[\rho_1'' - \rho_1 - 2\alpha_1' - 1\right] = -\left(\frac{GM}{R_1^3} - \frac{Gm}{d^3}\left[\frac{R_2-R_1}{R_1}\right]\right) + 2\rho_1\left(\frac{GM}{R_1^3} - \frac{Gm}{d^3}\left[\frac{R_2-R_1}{R_1}\right]\right)$$
$$- \frac{2Gm}{d^3}\frac{R_2}{R_1}\left(\rho_2 - \rho_1\right) \tag{44}$$

The relation for ω^2, Eq. (41), permits this result to be simplified to the following form if $Gm/(\omega^2 d^3) = \delta_1$ and $(R_1/R_2) = \delta_2$,

$$\rho_1'' - 3\rho_1 - 2\alpha_1' = -2(\delta_1/\delta_2)(\rho_2 - \rho_2) \ . \tag{45}$$

The second dynamical equation can easily be simplified to the form

$$\alpha_1'' + 2\rho_1' = (\delta_1/\delta_2)(\alpha_2 - \alpha_1) \ . \tag{46}$$

The remaining pair of dynamical equations can be simplified in the same way if the alternative expression for ω^2, Eq. (42), is used. The four equations can be written as

$$\begin{aligned}
\rho_1'' - 3\rho_1' - 2\alpha_1' &= -2(\delta_1/\delta_2)(\rho_2 - \rho_1) = -3(\rho_2 - \rho_1) \\
\alpha_1'' + 2\rho_1' &= (\delta_1/\delta_2)(\alpha_2 - \alpha_1) = (3/2)(\alpha_2 - \alpha_1) \\
\rho_2'' - 3\rho_2 - 2\alpha_2' &= 2(\delta_1 \ \delta_2)(\rho_2 - \rho_1) = 3(\rho_2 - \rho_1) \\
\alpha_2'' + 2\rho_2' &= (\delta_1 \ \delta_2)(\alpha_2 - \alpha_1) = (3/2)(\alpha_2 - \alpha_1)
\end{aligned} \tag{47}$$

where the second set of right hand sides are approximations to those which precede them since $(R_2/R_1) = \delta_2$ is very close to unity and $(Gm)/(\omega^2 d^3) = \delta_1$ is nearly $3/2$ under the same assumptions. If the simplified forms are taken the above set of four equations can be decomposed into the following two pairs of equations by straightforward additions and subtractions. The first pair of

$$\left.\begin{aligned}(\rho_2+\rho_1)'' - 3(\rho_2+\rho_1) - 2(\alpha_2+\alpha_1)' &= 0\\ (\alpha_2+\alpha_1)'' + 2(\rho_2+\rho_1)' &= 0\end{aligned}\right\}, \tag{48}$$

$$\left.\begin{aligned}(\rho_2-\rho_1)'' - 9(\rho_2-\rho_1) - 2(\alpha_2-\alpha_1)' &= 0\\ (\alpha_2-\alpha_1)'' + 3(\alpha_2-\alpha_1) + 2(\rho_2-\rho_1)' &= 0\end{aligned}\right\}, \tag{49}$$

equations relates to the motion of the mass-center of the pair of particles and the second pair to the relative motion of the particles. To determine the solutions let

$$\rho_1+\rho_2 = Ae^{\mu\tau}, \qquad \alpha_1+\alpha_2 = B\epsilon^{\mu\tau},$$

and

$$\rho_1-\rho_2 = C\epsilon^{\mu\tau}, \qquad \alpha_1-\alpha_2 = De^{\mu\tau},$$

in the two sets of equations respectively. The two characteristic equations are

$$\begin{vmatrix}(\mu^2-3) & -2\mu\\ 2\mu & \mu^2\end{vmatrix} = 0 \text{ and } \begin{vmatrix}(\nu^2-9) & -2\nu\\ 2\nu & (\nu^2+3)\end{vmatrix} = 0,$$

or

$$\mu^2 = 0,\ -1 \qquad \text{and} \qquad \nu^2 = 1 \pm \sqrt{28}\ . \tag{50}$$

Thus the disturbed motion is unstable both with respect to the motion of the mass-center and the relative motion with respect to the mass-center. The complete solution for all four quantities can now easily be carried out, but more direct interest is in the solution for the relative motion; this is the motion which characterizes the separation of the particles. Here we have the relations $C(\nu^2 - 9) = (2\nu)D$ for the coefficients. If $\nu^2 = 1 + \sqrt{28}$, $\nu = \pm 2.51$ and $C/D = \mp 1.86$; if $\nu^2 = 1 - \sqrt{28}$, $\nu = \pm i2.08$ and $C/D = \mp i0.156$. If new coefficients are introduced as

$$D_1 = (E_1+E_2)/2, \quad D_2 = (E_1-E_2)/2 \quad \text{and} \quad D_3 = (E_3+iE_4)/(2i) \ , \quad D_4 = (E_3-iE_4)/(2i) \ ,$$

the solutions for the motion of separation can be written as

$$\alpha_2 - \alpha_1 = E_1 \cosh(2.51\tau) + E_2 \sinh(2.51\tau) + E_3 \cos(2.08\tau) + E_4 \sin(2.08\tau), \quad (51)$$

$$\rho_2 - \rho_1 = -1.86E_1 \sinh(2.51\tau) - 1.86E_2 \cosh(2.51\tau) - 0.156E_3 \sin(2.08\tau) - 0.156E_4 \cos(2.08\tau) \ , \quad (51)$$

and the solutions for the motion of the mass-center can be written as

$$\begin{aligned} \alpha_2 + \alpha_1 &= (F_1 + F_2\tau) + F_3\cos(\tau) + F_4\sin(\tau) \ , \\ \rho_2 + \rho_1 &= (2/3)F_2 + (1/2)F_3\sin(\tau) - (1/2)F_4\cos(\tau) \ , \end{aligned} \quad (52)$$

in terms of new constants; there are 8 arbitrary constants since the position and velocity of each particle can be prescribed. At $t = 0$ all 4 coordinates may be assumed to be equal to zero, that is, $\alpha_2 \pm \alpha_1 = 0$ and $\rho_2 \pm \rho_1 = 0$ and only 4 constants remain in the solutions:

$$\alpha_2 - \alpha_1 = E_1(\cosh 2.51\tau - \cos 2.08\tau) + E_2(\sinh 2.51\tau - 11.9 \sin 2.08\tau) ,$$

$$\rho_2 - \rho_1 = E_1(-1.86 \sin 2.51\tau + 0.156 \sin 2.08\tau) + 0.186 E_2(-\cosh 2.51\tau + \cos 2.08\tau) ,$$

$$\alpha_2 + \alpha_1 = F_1(1 - \cos \tau) + F_2(\tau - (4/3) \sin \tau) , \tag{53}$$

$$\rho_2 + \rho_1 = -(2/3) F_2(1 - \cos \tau) - (1/2) F_1 \sin \tau .$$

The equations may be solved under various initial conditions.

For example we may set $(\rho_2 - \rho_1)' = 0$ and $(\alpha_2 - \alpha_1)' = q_o$ and determine the time $\tau = \tau^*$ at which $(\alpha_2 - \alpha_1)$ vanishes. This requirement leads to the equations

$$\alpha_2 - \alpha_1 = 0.044 q_o [\sinh 2.51\tau - 11.9 \sin 2.08\tau] , \tag{54}$$

(54)

$$\rho_2 - \rho_1 = 0.186(0.044 q_o) [-\cosh 2.51\tau + \cos 2.08\tau] ,$$

and if $\alpha_2 = \alpha_1$ at $\tau = \tau^*$, $(\rho_2 - \rho_1) = p_o$, a given number, then

$$\sinh 2.51\tau^* = 11.9 \sin 2.08\tau^* .$$

The value of τ^* is uniquely determined from this equation and p_o corresponds to the condition $r_1 = r_2$. Thus the second equation of Eq. (54) fixes the value of q_o for this motion to take place. That is, q_o is now fixed so that at $\tau = \tau^*$ the angle φ will be zero and the distance of separation will be zero; in order for this to hold under the assumed initial conditions, only one value of q_o is possible.